

A NEW CLASS OF SURFACES WITH MAXIMAL PICARD NUMBER

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ABSTRACT. A new class of examples of surfaces with maximal Picard number is constructed. These carry pencils of genus two or three curves such that their Jacobian fibrations are isogenous to fibre products of elliptic modular surfaces.

Let us say that a smooth complex projective surface is Picard maximal if the Picard number ρ equals the Hodge number $h^{1,1}$; in other words, if ρ is as large as possible. It is easy to see that any surface with trivial geometric genus is Picard maximal, but finding examples with $p_g > 0$ is much subtler. Picard maximal abelian or K3 surfaces can be constructed by taking a product of a CM elliptic curve with itself or the associated Kummer surface. Shioda [Sh] showed that all elliptic modular surfaces are Picard maximal. The first published examples of general type go back to Persson [P], who used double covers branched over rather special configurations of curves. A few more examples have since been found, and we refer to the Beauville's article [B] for a survey. The goal of this note is give some new, and we believe rather natural, examples, most of which have general type. The inspiration for us came from Shioda's work [Sh] mentioned above. Our examples, which carry pencils of genus two or three curves, are related in the sense that the Jacobian fibrations are isogenous to fibre products of elliptic modular surfaces. This is the key point that makes the examples work. It is worth remarking that these examples are defined over \mathbb{Q} , and they give new examples where Tate's conjecture holds.

1. HODGE THEORY OF FIBERED SURFACES

Let X be a smooth complex projective surface. Our interest is in $H^2(X)$. This carries a canonical weight 2 Hodge structure with respect to which the Neron-Severi group $NS(X)$ can be identified with $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ by Lefschetz's theorem. This yields the well known inequality $\rho(X) = \text{rank } NS(X) \leq h^{1,1}(X)$. We will say that X is *Picard maximal*, if equality holds. Part of the interest in this class stems from the following observation of Faltings [T, pp 81-82]: If X is defined over a finitely generated subfield $k \subset \mathbb{C}$ and $X_{\mathbb{C}}$ is Picard maximal, then Tate's conjecture holds for X , i.e. Galois invariant part of étale cohomology $H_{\text{ét}}^2(X_{\bar{k}}, \mathbb{Q}_{\ell}(1))^{Gal(\bar{k}/k)}$ is generated by divisors.

Now suppose that X carries a surjective holomorphic map $f : X \rightarrow C$ with connected fibres to a smooth projective curve. We suppose also that f possesses a section $\sigma : C \rightarrow X$. Let g be the genus of the general fibres of f and let q be the genus of C . Also let $j : U \rightarrow C$ be a nonempty Zariski open set over which $X^o = f^{-1}U \rightarrow U$ is smooth. We can analyze $H^2(X)$ using either the Leray

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spectral sequence or the decomposition theorem. We use the latter since it is bit more convenient. By Saito's version of the decomposition theorem [Sa, p 857], we can decompose $\mathbb{R}f_*\mathbb{Q}$ as a sum of intersection cohomology complexes up to shift in the constructible derived category, and moreover these complexes underly pure Hodge modules. By restricting this sum to U and applying Deligne's theorem [D1], we can identify some of these components explicitly:

$$(1) \quad \mathbb{R}f_*\mathbb{Q} \cong \underbrace{\mathbb{Q}}_{f_*\mathbb{Q}} \oplus j_*j^*R^1f_*\mathbb{Q}[-1] \oplus \underbrace{\mathbb{Q}}_{j_*j^*R^2f_*\mathbb{Q}}[-2] \oplus M$$

The, as yet undetermined, term M is supported on the finite set $S = C - U$. This yields a (noncanonical) decomposition

$$(2) \quad H^2(X, \mathbb{Q}) \cong f^*H^2(C, \mathbb{Q}) \oplus IH^1(R^1f_*\mathbb{Q}) \oplus H^0(C, j_*j^*R^2f_*\mathbb{Q}) \oplus H^2(M)$$

where $IH^1(R^1f_*\mathbb{Q}) = H^1(j_*j^*R^1f_*\mathbb{Q})$. The first summand on the right is spanned by the fundamental class $[X_t]$ of a fibre. The third summand is spanned by the class $[\sigma]$. To calculate M , we restrict to a point $s \in S$, and observe that $H^*(\mathbb{R}f_*\mathbb{Q}|_s)$ is the cohomology of the fibre $X_s = f^{-1}(s)$ by proper base change [Di, p 41]. Therefore M gives the excess cohomology not coming from the preceding terms in (1). Let

$$X_s = \sum_{i=1}^{m_s} n_{s,i} X_{s,i}$$

be the decomposition into irreducible components. Let D_s be a small disk centered at s , $t \in D_s^* = D_s - \{s\}$, and $\gamma_s \in \pi_1(D_s^*, t)$ a generator. Then after combining the local invariant cycle theorem [Sc] with some elementary topological arguments, we see that

$$H^i(X_s, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ H^1(X_t, \mathbb{Q})^{\gamma_s} \cong (j_*j^*R^1f_*\mathbb{Q})_s & \text{if } i = 1 \\ \mathbb{Q}^{m_s} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore $M \cong \bigoplus \mathbb{Q}_s^{m_s-1}[-2]$. From (2), we deduce that we have a noncanonical decomposition

$$(3) \quad H^2(X) \cong IH^1(R^1f_*\mathbb{Q}) \oplus \mathbb{Q}[\sigma] \oplus \mathbb{Q}[X_t] \oplus \bigoplus_s \mathbb{Q}^{m_s-1}$$

We can see that the last two summands are spanned by divisor classes supported on the fibres. Since the decomposition (1) can be lifted to the derived category of Hodge modules, we see that (3) becomes an isomorphism of Hodge structures provided that all the summands in (3) except the first are viewed as sums of the Tate structures $\mathbb{Q}(-1)$. The first summand $IH^1(R^1f_*\mathbb{Q})$ is the interesting piece. We note that an intrinsic Hodge structure on it was first constructed by Zucker [Z]. Since all but the first summand on the right of (3) are spanned by divisor classes, we may conclude that:

Proposition 1.1. *If $f : X \rightarrow C$ is a map satisfying the above assumptions, then X is Picard maximal if and only if $IH^1(R^1f_*\mathbb{C})^{(1,1)}$ is spanned by divisors. In particular, this is the case if $IH^1(R^1f_*\mathbb{C})^{(1,1)} = 0$.*

Although the decomposition (3) is not canonical, we note that $IH^1(R^1f_*\mathbb{Q})$ is the canonical subquotient L^1/L^2 of $H^2(X)$, where L^\bullet is filtration associated to the Leray spectral sequence. In particular, given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ & \searrow f' & \downarrow f \\ & & C \end{array}$$

the map $g^* : H^2(X) \rightarrow H^2(X')$ will take $IH^1(R^1f_*\mathbb{Q})$ to $IH^1(R^1f'_*\mathbb{Q})$.

We recall that the Mordell-Weil group $MW(X/C)$ is the group of sections of the associated Jacobian fibration $Pic^0(X^o/U)$. The group is finitely generated if $Pic^0(X^o/U)$ has no fixed part, i.e. no nonzero constant abelian subschemes [C]. We will say that a surface satisfying $IH^1(R^1f_*\mathbb{C})^{(1,1)} = 0$ is *extremal*. While this terminology is not very descriptive, it conforms to standard usage in elliptic surface theory [M, p 75] because of the following:

Lemma 1.1. *Suppose that the Jacobian fibration associated to X^o/U has no fixed part. If X/C is extremal then the rank of $MW(X/C)$ is zero. The converse holds when X is Picard maximal.*

Proof. Formula (3) implies

$$\dim IH^1(R^1f_*\mathbb{C})^{(1,1)} = h^{1,1}(X) - 2 - \sum (m_s - 1)$$

The lemma is consequence of this together with the Shioda-Tate formula [Sh2, (4)]

$$\text{rank } MW(X/C) = \rho(X) - 2 - \sum (m_s - 1)$$

□

Given a finite index subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, we have an associated modular curve U_Γ given as a quotient of the upper half plane by Γ . Let C_Γ be the nonsingular compactification. We can interpret U_Γ as the moduli space of elliptic curves with some sort of level structure. The associated elliptic modular surface $\mathcal{E}_\Gamma \rightarrow C_\Gamma$ is the universal family of elliptic curves over U_Γ (minus the set of elliptic fixed points) suitably extended to C_Γ . As Shioda observed [Sh, 7.8] such surfaces are Picard maximal. In fact, the stronger property also holds.

Theorem 1.1 (Shioda). *Suppose that $-I \notin \Gamma$ and let $f : \mathcal{E}_\Gamma \rightarrow C_\Gamma$ be corresponding elliptic modular surface. Then $\mathcal{E}_\Gamma \rightarrow C_\Gamma$ is extremal.*

Proof. This is proved in [Sh, 4.12]. (Shioda formulates this using Kodaira's homological invariant G , but this can be identified with $j_*j^*R^1f_*\mathbb{Z}$.) □

Remark 1.1. *The conclusion also applies to the Legendre family $y^2 = x(x-1)(x-\lambda)$, which is the elliptic modular surface for $\Gamma(2)$, even though the theorem does not. The point is this surface is rational and therefore Picard maximal. Furthermore, Igusa [I] has shown that the rank of the Mordell-Weil group is zero.*

2. FREY-KANI CONSTRUCTION

Given an integer $n > 0$, by degree n FK data, we will mean the following: a pair of elliptic curves E, E' , and an isomorphism of n -torsion subgroups $\phi : E[n] \rightarrow E'[n]$ which is an anti-isometry with respect to the Weil pairings. Let $\Gamma_\phi \subset (E \times E')[n]$ be subgroup given by the graph of ϕ .

Theorem 2.1 (Frey-Kani). *Given degree n FK data, the abelian surface $J = E \times E' / \Gamma_\phi$ carries a unique principal polarization Θ such that $\pi^* \Theta \sim n(E \times 0 + 0 \times E')$ where $\pi : E \times E' \rightarrow J$ is the projection. Either*

- (1) Θ is an irreducible smooth curve of genus 2, or
- (2) Θ is a sum of two elliptic curves meeting at one point.

The second case can only happen if E and E' are isogenous.

Proof. This was proved in [FK] with some additional restrictions. These were relaxed in [K, 1.5, 2.3]. \square

If case (1) holds above, we will say that the FK data is *irreducible*, or that the anti-isometry ϕ is *irreducible*.

Theorem 2.2 (Kani). *If n is prime, then*

- (1) *There exists an irreducible anti-isometry $\phi : E[n] \rightarrow E'[n]$ for any pair of elliptic curves E, E' .*
- (2) *An anti-isometry $\phi : E[n] \rightarrow E'[n]$ is reducible if and only if there is an isogeny $h : E \rightarrow E'$ of degree $k(n - k)$ for some $1 \leq k < n$, such that*

$$\phi \circ [k] = h|_{E[n]}.$$

Proof. This was proved in [K, Theorems 2 and 3]. \square

Corollary 2.3. *Let E be an elliptic curve without complex multiplication and let n be a prime such that $n \equiv 3 \pmod{4}$. Then all the anti-isometries $\phi : E[n] \rightarrow E'[n]$ are irreducible.*

Proof. We prove that there are no isogenies $h : E \rightarrow E$ of degree $k(n - k)$ for any $1 \leq k < n$. If $h : E \rightarrow E$ is an isogeny, then $\deg(h) = m^2$ for some integer m , since E does not have complex multiplication. But m^2 cannot equal $k(n - k)$, since otherwise -1 would be a quadratic residue modulo n . \square

Corollary 2.4. *Let E be any elliptic curve such that $j(E) \neq 0, 1728$. Then all the anti-isometries $\phi : E[2] \rightarrow E[2]$ which are not equal to the identity are irreducible.*

Proof. For $n = 2$, anti-isometries are exactly the same as isometries. Since E has no automorphisms other than $\pm \text{id}_E$ when $j(E) \neq 0, 1728$, all the anti-isometries which are not the identity are irreducible. \square

Remark 2.1. *The Θ -divisors associated to ϕ and $-\phi$ are the same as abstract curves. Therefore ϕ is irreducible if and only if $-\phi$ is. Thus we may extend this terminology to orbits $\{\pm\phi\}$.*

3. THE EXAMPLES

Let $G = SL_2(\mathbb{Z}/n\mathbb{Z})$ and let $\Gamma(n) = \ker[SL_2(\mathbb{Z}) \rightarrow G]$ be the principal congruence group of level n . Let $f : \mathcal{E} = \mathcal{E}_{\Gamma(n)} \rightarrow C_{\Gamma(n)} = C$ be the associated elliptic modular surface, and let $U \subset C$ be the complement of the discriminant. The group G acts on C through the quotient $\bar{G} = PSL_2(\mathbb{Z}/n\mathbb{Z}) = \text{Gal}(C/C_{\Gamma(1)})$. Let $f_\sigma : \mathcal{E}_\sigma \rightarrow C$ be the pullback of $\mathcal{E} \rightarrow C$ along $\sigma : C \rightarrow C$. This only depends on the image $\bar{\sigma} = \text{im } \sigma \in \bar{G}$, so we may also denote this by $f_{\bar{\sigma}} : \mathcal{E}_{\bar{\sigma}} \rightarrow C$. Fix $t \in U$ and a reference anti-isometry $\phi_1 : (\mathbb{Z}/n\mathbb{Z})^2 \rightarrow (\mathbb{Z}/n\mathbb{Z})^2$ represented by say $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Observe that the fibres $E = \mathcal{E}_t$ and $E' = \mathcal{E}_{\sigma,t}$ are the same elliptic curve equipped with different level n structures $(\mathbb{Z}/n\mathbb{Z})^2 \cong E[n]$ and $(\mathbb{Z}/n\mathbb{Z})^2 \cong E'[n]$. Thus ϕ_1 induces an anti-isometry $\phi_\sigma : E[n] \cong E'[n]$.

We need to make the construction of ϕ_σ a bit more precise. Let $\tilde{f} : \tilde{\mathcal{E}} \rightarrow H$ be the universal marked elliptic curve over the upper half plane. By “marked”, we mean that there is fixed symplectic isomorphism $\lambda : R^1 \tilde{f}_* \mathbb{Z} \cong \mathbb{Z}^2$, where the right side is equipped with the standard pairing represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The group $SL_2(\mathbb{Z})$ acts equivariantly on \tilde{f} , and $R^1 \tilde{f}_* \mathbb{Z}$ is the equivariant constant sheaf associated to the standard representation of this group. More concretely, this means that for $\bar{\sigma} \in SL_2(\mathbb{Z})$, the base change map $\bar{\sigma}^* R^1 \tilde{f}_* \mathbb{Z} \rightarrow R^1 \tilde{f}_* \mathbb{Z}$ corresponds, under λ , to multiplication by $\bar{\sigma}$ on \mathbb{Z}^2 . Given a preimage $\bar{\sigma} \in SL_2(\mathbb{Z})$ of $\sigma \in G$, we have a commutative diagram

$$\begin{array}{ccccc}
 & & \tilde{\mathcal{E}} & \xrightarrow{\bar{\sigma}} & \tilde{\mathcal{E}} \\
 & \swarrow & \downarrow \tilde{f} & \searrow & \downarrow \tilde{f} \\
 \mathcal{E}_\sigma & \xrightarrow{\tilde{f}} & \mathcal{E} & & \mathcal{E} \\
 \downarrow f_\sigma & & \downarrow f & & \downarrow f \\
 H & \xrightarrow{f} & H & & H \\
 \swarrow \pi & & \downarrow \pi & \searrow \pi & \\
 U & \xrightarrow{\sigma} & U & & U
 \end{array}$$

where for simplicity of notation we have omitted the restriction symbols “ $|_{f^{-1}U}$ ”. With a little bit of thought, one sees that all the vertical squares are Cartesian. We can deduce from this, and the fact that $U = H/\Gamma(n)$, that λ modulo n descends to isomorphisms $\lambda_\sigma : R^1 f_{\sigma*} \mathbb{Z}/n\mathbb{Z}|_U \cong (\mathbb{Z}/n\mathbb{Z})^2$ for each σ . Furthermore, we get a commutative diagram

$$\begin{array}{ccc}
 (\mathbb{Z}/n\mathbb{Z})^2 & \xleftarrow{\lambda_1} & \pi^* R^1 f_* \mathbb{Z}/n\mathbb{Z}|_U \\
 \downarrow \sigma & & \downarrow \Lambda_\sigma \\
 (\mathbb{Z}/n\mathbb{Z})^2 & \xrightarrow{\lambda_\sigma^{-1}} & \pi^* R^1 f_{\sigma*} \mathbb{Z}/n\mathbb{Z}|_U
 \end{array}$$

of constant local systems which can be descended to U . Composing Λ_σ with the reference anti-isometry ϕ_1 gives a new anti-isometry $\phi_\sigma : R^1 f_* \mathbb{Z}/n\mathbb{Z}|_U \rightarrow R^1 f_{\sigma*} \mathbb{Z}/n\mathbb{Z}|_U$. This can be viewed as an anti-isometry from $\mathcal{E}[n]|_U \rightarrow \mathcal{E}_\sigma[n]|_U$,

thanks to the canonical isomorphisms $\mathcal{E}[n]|_U \cong R^1 f_* \mathbb{Z}/n\mathbb{Z}|_U$ and $\mathcal{E}'[n]|_U \cong R^1 f_{\sigma*} \mathbb{Z}/n\mathbb{Z}|_U$.

The set of anti-isometries of $(\mathbb{Z}/n\mathbb{Z})^2 \rightarrow (\mathbb{Z}/n\mathbb{Z})^2$ forms a torsor over G . In other words, all anti-isometries are given by composing ϕ_1 with an element of G . Therefore as σ varies in G we obtain all possible anti-isometries $\mathcal{E}_t[n] \rightarrow \mathcal{E}_{\sigma(t)}[n] = \mathcal{E}_t[n]$. We say that $\sigma \in G$ is irreducible if the FK data $(E = \mathcal{E}_t, E' = \mathcal{E}_{\sigma(t)}, \phi_\sigma)$ is irreducible. By the results of the previous section, all $\sigma \in G$ are irreducible when $n \equiv 3 \pmod{4}$ is a prime. For $n = 2$, all five $\sigma \in SL_2(\mathbb{Z}/2\mathbb{Z})$ such that $\sigma \neq \text{id}_2$ are irreducible. Moreover, when $n \equiv 1 \pmod{4}$ is a prime, there is at least one irreducible $\sigma \in G$.

Fix an irreducible element $\sigma \in G = SL_2(\mathbb{Z}/n\mathbb{Z})$. Let $X_{n,\sigma}^o \subset \mathcal{E} \times_U \mathcal{E}_\sigma / \Gamma_\phi$ be the relative Θ divisor with respect to a principal polarization satisfying the conditions of theorem 2.1 on the fibres. By assumption, some fibre of $X_{n,\sigma}^o \rightarrow C$ is a smooth irreducible curve of genus 2, therefore the same holds for all fibres over some Zariski open $V \subset U$ containing t . Let $F : X_{n,\sigma} \rightarrow C$ be a relatively minimal nonsingular compactification of the preimage of V in $X_{n,\sigma}^o$.

Theorem 3.1. *With the notation as in the last paragraph, $F : X_\sigma \rightarrow C$ is extremal, and therefore Picard maximal.*

Proof. Let us replace U by V . Since $\text{Pic}^0(X_{n,\sigma}^o/U) \rightarrow \mathcal{E} \times_U \mathcal{E}_\sigma$ is a fibrewise isogeny, $R^1 F_* \mathbb{Q}|_U \cong (R^1 f_* \mathbb{Q} \oplus R^1 f_{\sigma*} \mathbb{Q})|_U$ as Hodge modules. Note also that $R^1 f_{\sigma*} \mathbb{Q}|_U \cong \sigma^* R^1 f_* \mathbb{Q}|_U$. Therefore

$$IH^1(R^1 F_* \mathbb{C})^{(1,1)} = IH^1(R^1 f_* \mathbb{C})^{(1,1)} \oplus IH^1(R^1 f_* \mathbb{C})^{(1,1)} = 0$$

□

We mention a few related examples.

- (A) Choose a finite index subgroup $\Gamma \subseteq \Gamma(n)$. Let $X_{\Gamma,\sigma}$ be a minimal desingularization of the fibre product $X_{n,\sigma} \times_{C_{\Gamma(n)}} C_\Gamma$. By the same argument, we can see that $X_{\Gamma,\sigma} \rightarrow C_\Gamma$ is extremal, and consequently Picard maximal.
- (B) Let $f : X \rightarrow \mathbb{P}^1$ be obtained by blowing up \mathbb{P}^2 along the base locus of the pencil

$$\{x^4 + y^4 + z^4 + t(x^2 y^2 + y^2 z^2 + z^2 x^2) = 0\}_{t \in \mathbb{P}^1}$$

These curves are nonsingular when $t \notin S = \{-1, \pm 2, \infty\}$. Beauville [B, p 5] shows that when $t \notin S$, the curve $E_t = X_t / \{x \mapsto \pm x\}$ can be identified with the elliptic curve $y^2 = x(x-1)(x+t+1)$, and moreover the map induces an isogeny $\text{Pic}^0(X_t) \sim E_t^3$. In fact this is an isogeny of abelian schemes from $\text{Pic}^0(X/\mathbb{P}^1 - S)$ to $\mathcal{E}^3|_{\{\mathbb{P}^1 - S\}}$, where $g : \mathcal{E} \rightarrow C_{\Gamma(2)} = \mathbb{P}^1$ is the Legendre family pulled back along the automorphism that fixes 0, 1 and sends $t \mapsto -t - 1$. Thus

$$(4) \quad R^1 f_* \mathbb{Q} \cong (R^1 g_* \mathbb{Q})^3$$

and therefore $X \rightarrow \mathbb{P}^1$ is extremal. It follows that X is Picard maximal but this was already clear from the fact that it was rational. However, we can create a nonrational surface by choosing a subgroup $\Gamma \subseteq \Gamma(2)$ of sufficiently large finite index, and letting $X \rightarrow C_\Gamma$ be the desingularized pullback of S to C_Γ under the projection $C_\Gamma \rightarrow C_{\Gamma(2)} = \mathbb{P}^1$. The isomorphism (4) will

persist if we pull it back to C_Γ . Thus we see that X is also extremal and hence Picard maximal.

(C) Let E_t be the elliptic curve given by

$$y^2 = -\frac{t^3}{(1+t)^3}(x-1)(x+t)(x+1/t),$$

for $t \in U = \mathbb{P}^1 \setminus \{0, \pm 1, \infty\}$. Then E_t is isomorphic to the elliptic curve given by $y^2 = x(x-1)(x-t)$ as follows: The elliptic curve $y^2 = x(x-1)(x-t)$ is the double cover of \mathbb{P}^1 branched over $\{t, 1, 0, \infty\}$. If one pulls back this elliptic curve along the automorphism of \mathbb{P}^1 that sends

$$x \mapsto \frac{-t}{1+t}(x-1),$$

one gets E_t branched over $\{-t, -1/t, 1, \infty\}$. This works in families, and the family E_t is isomorphic to the Legendre family.

Now consider the family C_t of genus 2 curves given by

$$y^2 = -\frac{t^3}{(1+t)^3}(x^2-1)(x^2+t)(x^2+1/t).$$

These curves are nonsingular for $t \in U$. We have a surjective map $C_t \rightarrow E_t$ which sends $(x, y) \mapsto (x^2, y)$. Under this map, the differential $dx/2y \in H^0(E_t, \omega_{E_t})$ pulls back to $xdx/y \in H^0(C_t, \omega_{C_t})$. Consider the automorphism τ of C_t given by $x \mapsto 1/x$ and $y \mapsto y\sqrt{-1}/x^3$. We have $\tau^*(xdx/y) = \sqrt{-1}dx/y \in H^0(C_t, \omega_{C_t})$. Thus the two differential forms xdx/y and $\tau^*(xdx/y)$ give a basis for $H^0(C_t, \omega_{C_t})$. Now the proof of Lemma 2 in [B, p 4] gives an isogeny $\text{Pic}^0(C_t) \sim E_t^2$.

Let \mathcal{C} denote the family of the curves C_t as t varies in U , with the associated map $f : \mathcal{C} \rightarrow U$, and let \mathcal{E} denote the family of the curves E_t as t varies in U with the associated map $g : \mathcal{E} \rightarrow U$. The argument above works in a family and gives an isogeny of abelian schemes from $\text{Pic}^0(\mathcal{C}/U)$ to $\mathcal{E} \times_U \mathcal{E}$ for all $t \in U$. Then

$$R^1 f_* \mathbb{Q} = (R^1 g_* \mathbb{Q})^{\oplus 2},$$

and hence $\mathcal{C} \rightarrow U$ is extremal. Thus, \mathcal{C} is Picard maximal.

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